Math 335 Sample Problems

One notebook sized page of notes (*one side*)will be allowed on the test. No electronic devices allowed. You may work together on the sample problems – I encourage you to do that. The test will cover 4.5-4.7 and 5.2-5.8, 6.1. There may be homework problems on the test. The midterm is on Monday, February 3.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}^1$ be defined by

$$f(x,t) = \begin{cases} \frac{\sin(xt)}{t} \text{ if } t \neq 0, \\ x \text{ if } t = 0. \end{cases}$$

   Let

$$g(x) = \int_0^{\pi/2} f(x,t)dt.$$

   Compute $g'(x)$ and $g'(0)$.

2. (a) Compute $\displaystyle\int_{x^2+y^2=1} \frac{-ydx + xdy}{x^2+y^2}$.

   (b) Using part (a) and Green's theorem, compute $\displaystyle\int_{\frac{x^2}{4}+\frac{y^2}{9}=1} \frac{-ydx + xdy}{x^2+y^2}$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuously differentiable and assume $f(\theta) > 0$. Use Green's theorem to prove that area of the region S, defined in polar coordiates by the inequalities

$$\alpha \leq \theta \leq \beta, r \leq f(\theta),$$

   is given by

$$A(S) = \frac{1}{2}\int_\alpha^\beta f^2(\theta)d\theta.$$

4. Suppose f is continuous on $[0,\infty)$ and $|xf(x)| < 1$ for $x \geq 1$. Prove or give a counterexample to the statement that $\int_1^\infty f(x)dx$ converges.

5. Let C be the curve of intersection of $y + z = 0$ and $x^2 + y^2 = a^2$ oriented in the counterclockwise direction when viewed from a point high on the z-axis. Use Stokes' theorem to compute the value of $\displaystyle\int_C (xz+1)dx + (yz+2x)dy$.

6. Let
$$\phi(x) = \int_0^\pi \cos(x \sin t) dt.$$
Prove that
$$x\phi''(x) + \phi'(x) + x\phi(x) = 0.$$

7. (a) Prove that $\int_C \frac{-ydx + xdy}{x^2 + y^2}$ is not independent of path on $\mathbf{R}^2 - \mathbf{0}$.

   (b) Prove that $\int_C \frac{xdx + ydy}{x^2 + y^2}$ is independent of path on $\mathbf{R}^2 - \mathbf{0}$. Find a function $f(x, y)$ on $\mathbf{R}^2 - \mathbf{0}$ so that $\nabla f = (\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2})$.

8. Prove that $\int_0^\infty \cos x^2 dx$ converges, but not absolutely.

9. Decide if the following integrals converge conditionally, converge absolutely, or diverge.

   (a)
$$\int_{-\infty}^{+\infty} x^2 e^{-|x|} dx$$

   (b)
$$\int_0^\pi \frac{dx}{(\cos x)^{\frac{2}{3}}}$$

   (c)
$$\int_1^\infty \frac{\sin(1/x)}{x} dx$$

10. Let f and g be integrable on $[a, b]$ for every $b > a$.

    (a) Prove that
$$(\int_a^b |fg|)^2 \leq \int_a^b f^2 \int_a^b g^2.$$
    You must give a proof of this. It is not proved in the text.

    (b) Prove that if $\int_a^\infty f^2$ and $\int_a^\infty g^2$ converge then $\int_a^\infty fg$ converges absolutely.

11. Let $a_n = \log(\frac{n}{n+1})$. Does $a_n \to 0$? Does the series $\sum_1^\infty a_n$ converge? If so, find its limit.

12. Let S be the surface (torus) obtained by rotating the circle $(x - 2)^2 + z^2 = 1$ around the z-axis. Compute the integral $\int_S \mathbf{F} \cdot \mathbf{n} dA$, where $\mathbf{F} = (x + \sin(yz), y + e^{x+z}, z - x^2 \cos y)$.

13. Let $w(x)$ satisfy $w''(x) + w(x) = 0, w(0) = 0, w'(0) = 1$. Let $f(x) = \int_0^x (w(x-y))h(y)dy$. Prove that

$$f''(x) + f(x) = h(x), f(0) = 0, f'(0) = 0.$$

14. We have covered the following:
    (a) Green's theorem
    (b) Surface area
    (c) Divergence theorem
    (d) Stokes' theorem
    (e) Integrating vector derivatives
    (f) Integrals dependent on a parameter
    (g) Improper single and multiple integrals
    (h) Introduction to infinite series.

15. There may be homework problems or example problems from the text on the midterm.